# *Automatic Control (1)*

*By*

*Associate Prof. / Mohamed Ahmed Ebrahim Mohamed*

E-mail: mohamedahmed_en@yahoo.com

mohamed.mohamed@feng.bu.edu.eg

Web site: http://bu.edu.eg/staff/mohamedmohamed033

# Lecture (4)

# Block Diagram Reduction using Decomposition

# &

# Time Domain Analysis

# Time Domain Analysis

*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

# Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

- Usually, the input signals to control systems are not known fully ahead of time.

- It is therefore difficult to express the actual input signals mathematically by simple equations.

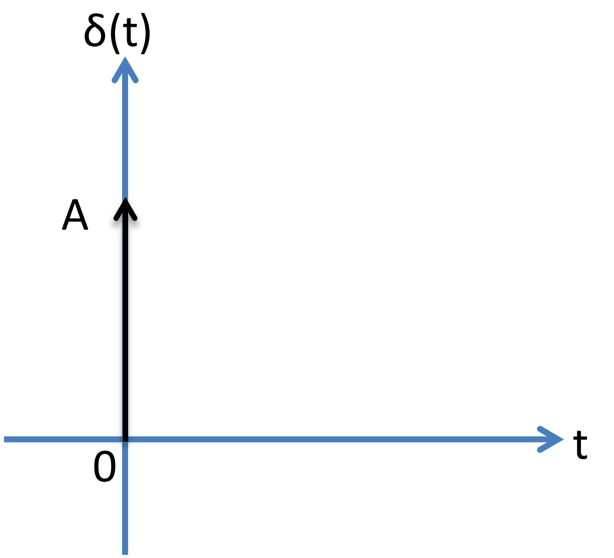
# Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.
- The other standard signal of great importance is a sinusoidal signal.

# Standard Test Signals

- Impulse signal
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

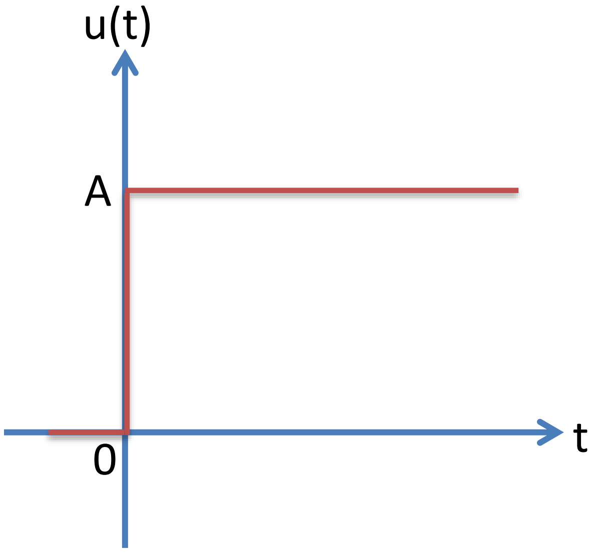



  - If A=1, the impulse signal is called unit impulse signal.

# Standard Test Signals

- Step signal
  - The step signal imitate the sudden change characteristic of actual input signal.

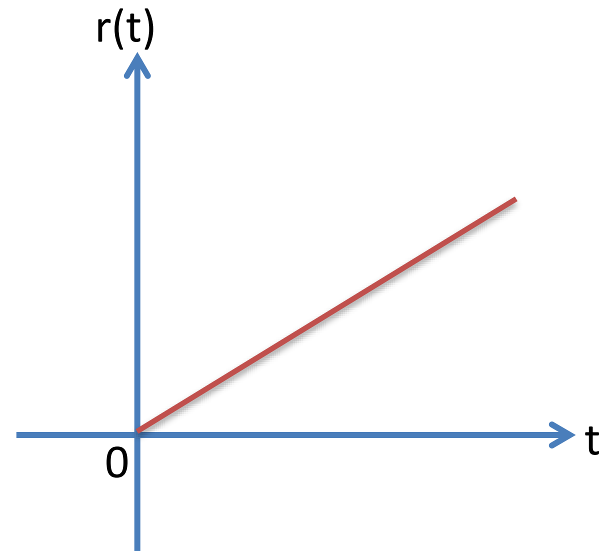
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



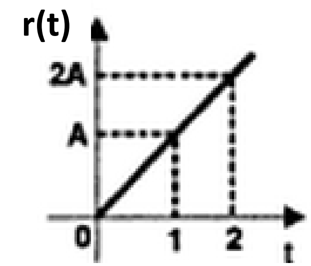

  - If A=1, the step signal is called unit step signal

# Standard Test Signals

- Ramp signal
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

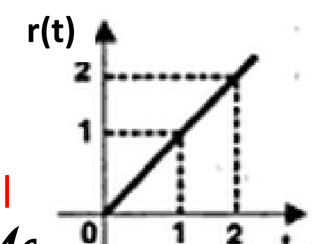



$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$




ramp signal with slope A

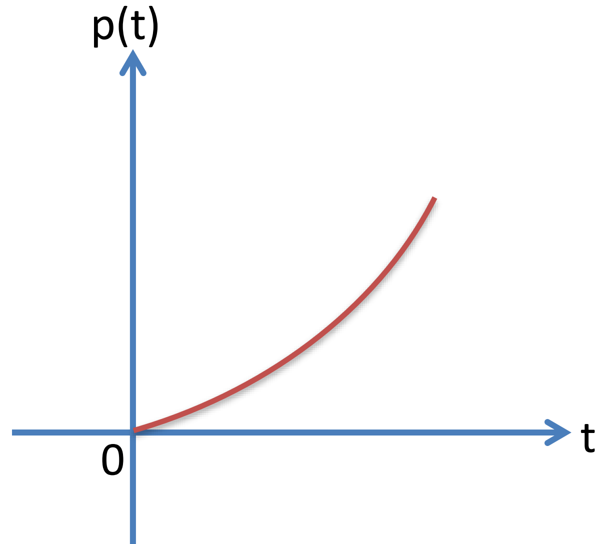
  - If *A=1*, the ramp signal is called unit ramp signal



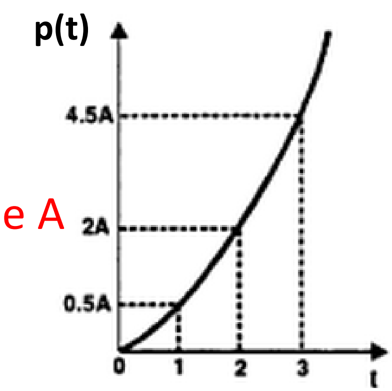

unit ramp signal

# Standard Test Signals

- Parabolic signal
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

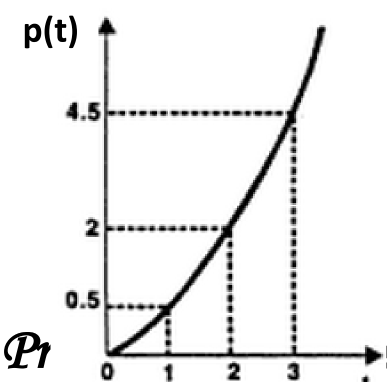



$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$




parabolic signal with slope A

  - If *A=1*, the parabolic signal is called unit parabolic signal.




Unit parabolic signal

# Relation between standard Test Signals

- Impulse

$$\delta(t)=\begin{cases}A & t=0\\ 0 & t\neq 0\end{cases}$$

$$\int \qquad \frac{d}{dt}$$

- Step

$$u(t)=\begin{cases}A & t\geq 0\\ 0 & t<0\end{cases}$$

$$\int \qquad \frac{d}{dt}$$

- Ramp

$$r(t)=\begin{cases}At & t\geq 0\\ 0 & t<0\end{cases}$$

$$\int \qquad \frac{d}{dt}$$

- Parabolic

$$p(t)=\begin{cases}\dfrac{At^2}{2} & t\geq 0\\ 0 & t<0\end{cases}$$

*Associate Prof. Dr. Mohamed Ahmed Ebrahim*

# Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

# Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

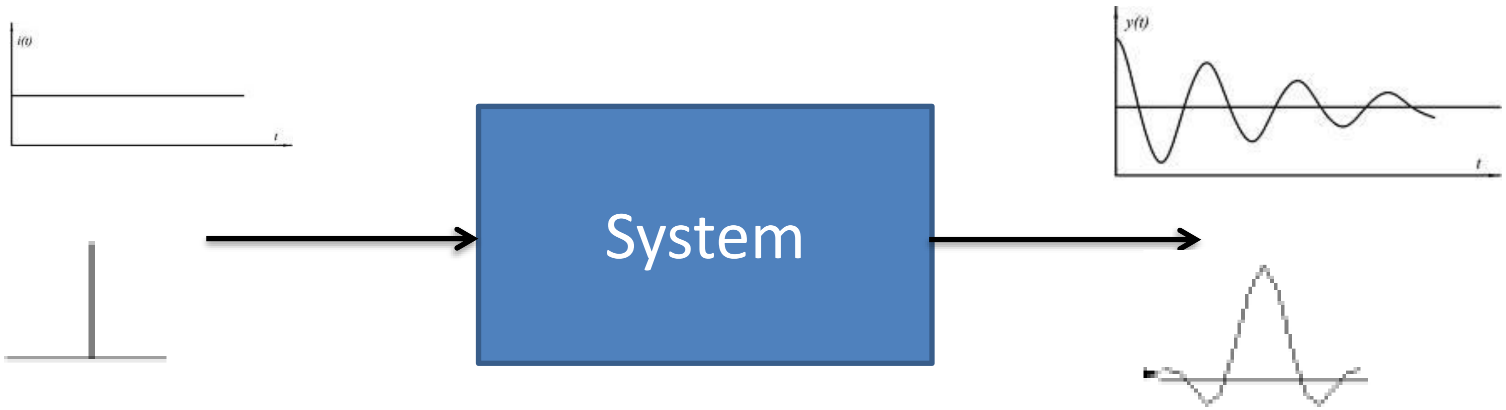
- Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

# Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.




- The time response of any system has two components
  - Transient response
  - Steady-state response.

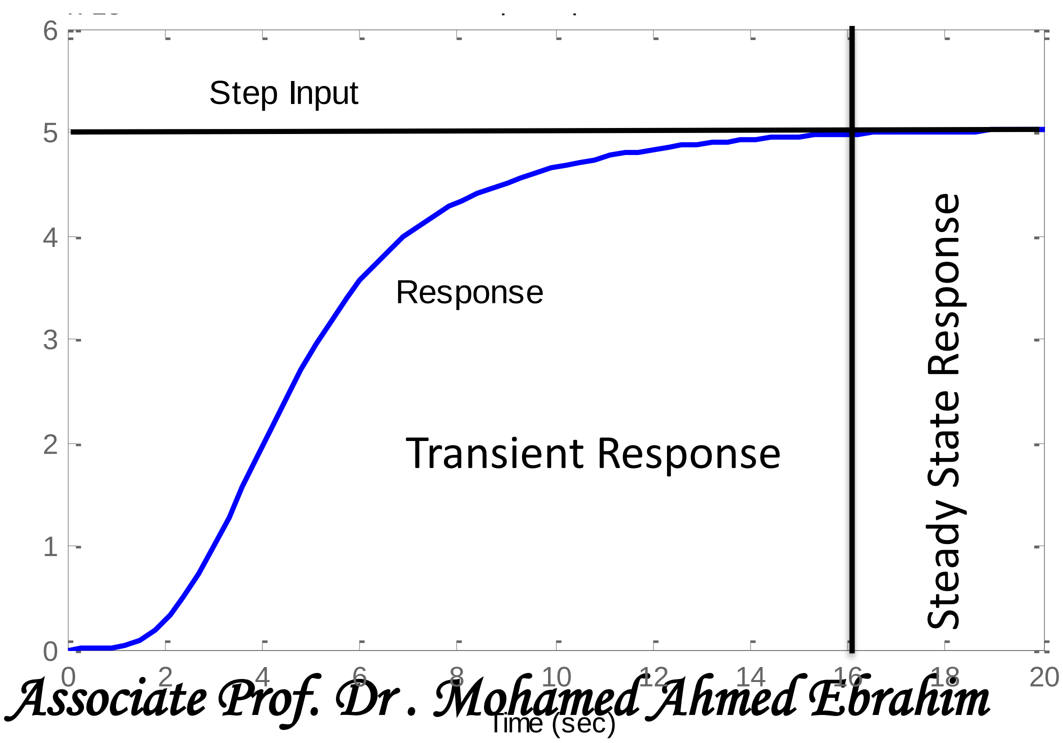
*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

# Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.




Associate Prof. Dr . Mohamed Ahmed Ebrahim

# Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final value theorem.

# Introduction

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

# Introduction

- The first order system given below.

$$G(s) = \frac{10}{5s + 1}$$

- D.C gain is 10 and time constant is 5 seconds.

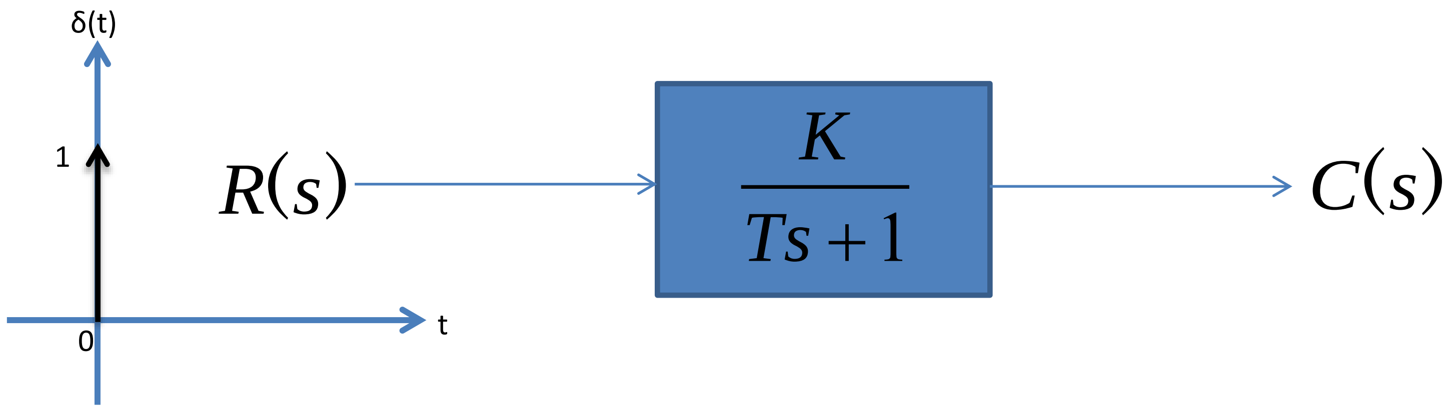
- For the following system

$$G(s) = \frac{6}{s + 2} \qquad = \frac{6/2}{1/2s + 1}$$

- D.C Gain of the system is 6/2 and time constant is 1/2 seconds.

# Impulse Response of 1st Order System

- Consider the following 1st order system




$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

# Impulse Response of 1st Order System

$$C(s)=\frac{K}{Ts+1}$$

- Re-arrange following equation as

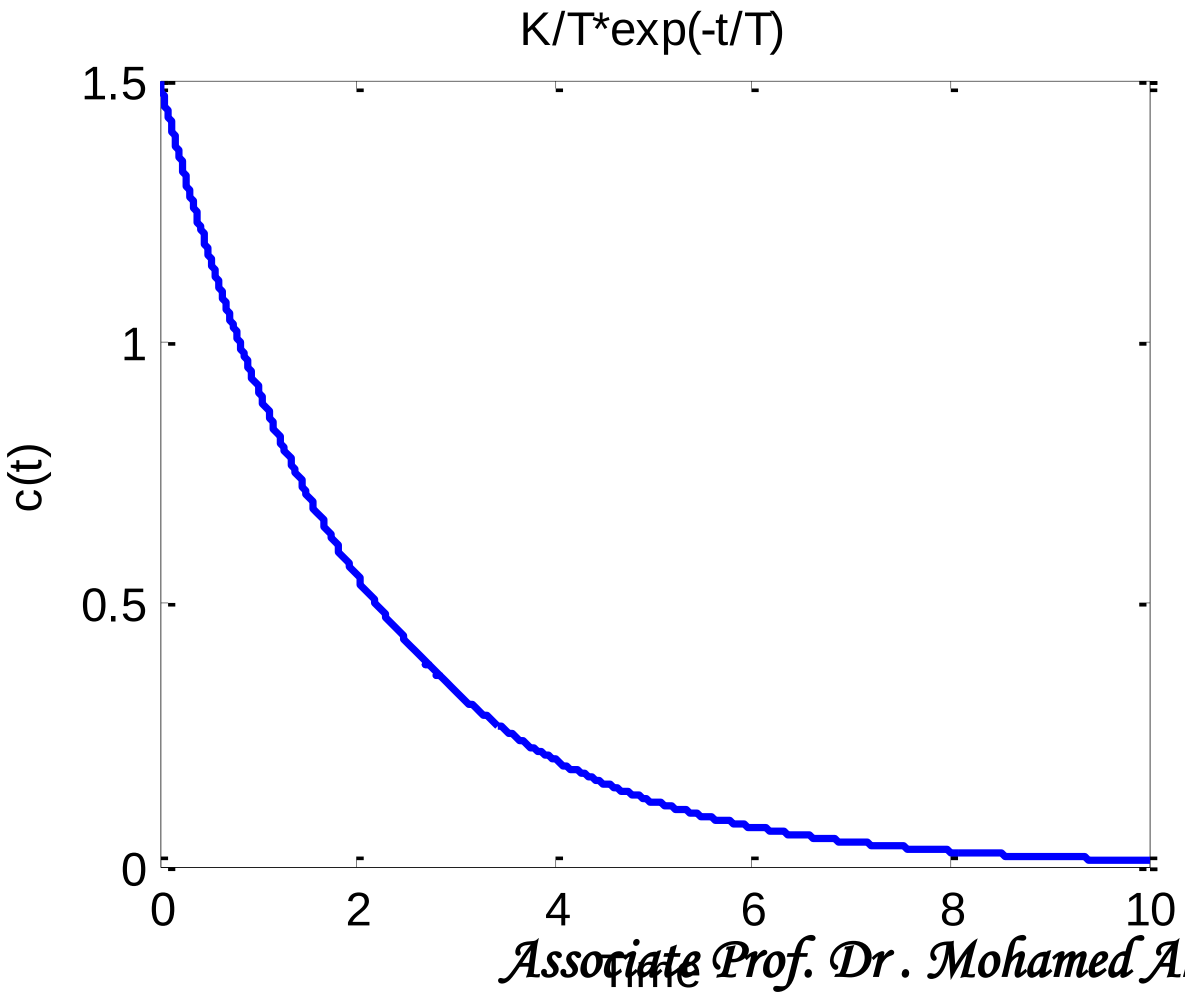
$$C(s)=\frac{K/T}{s+1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right)=Ce^{-at} \qquad c(t)=\frac{K}{T}e^{-t/T}$$

# Impulse Response of 1st Order System

- If K=3 and T=2s then $c(t) = \frac{K}{T} e^{-t/T}$



# Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion.

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

# Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$

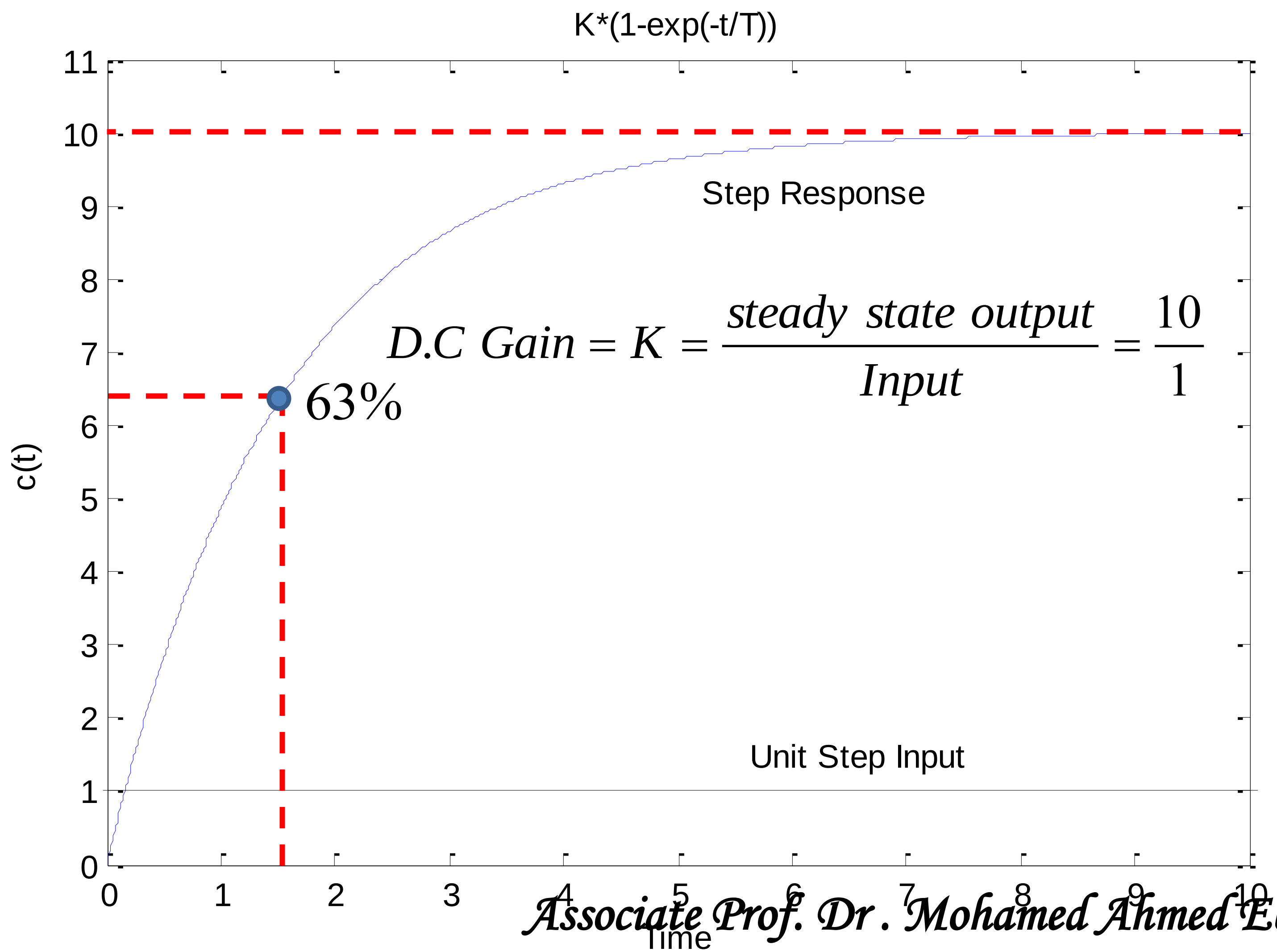
- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

# Step Response of 1st Order System

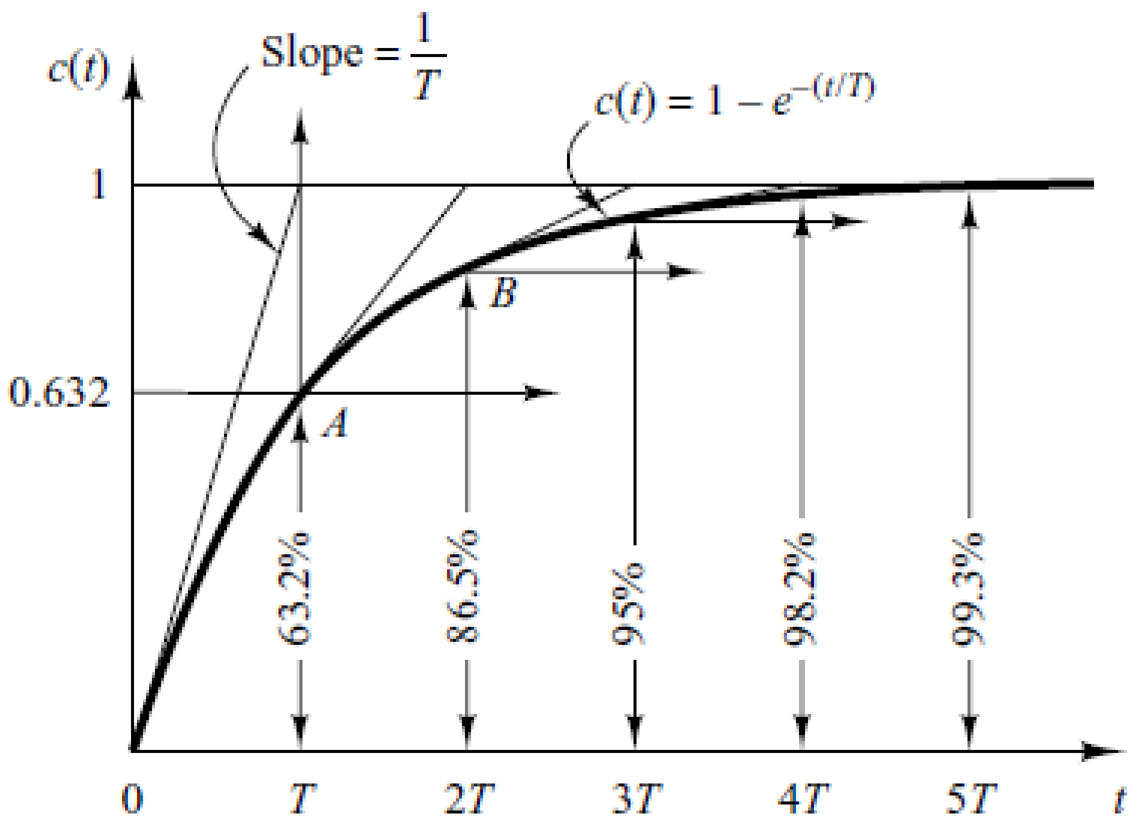
- If K=10 and T=1.5s then $$c(t) = K\left(1 - e^{-t/T}\right)$$




Associate Prof. Dr. Mohamed Ahmed Ebrahim

# Step Response of 1st order System

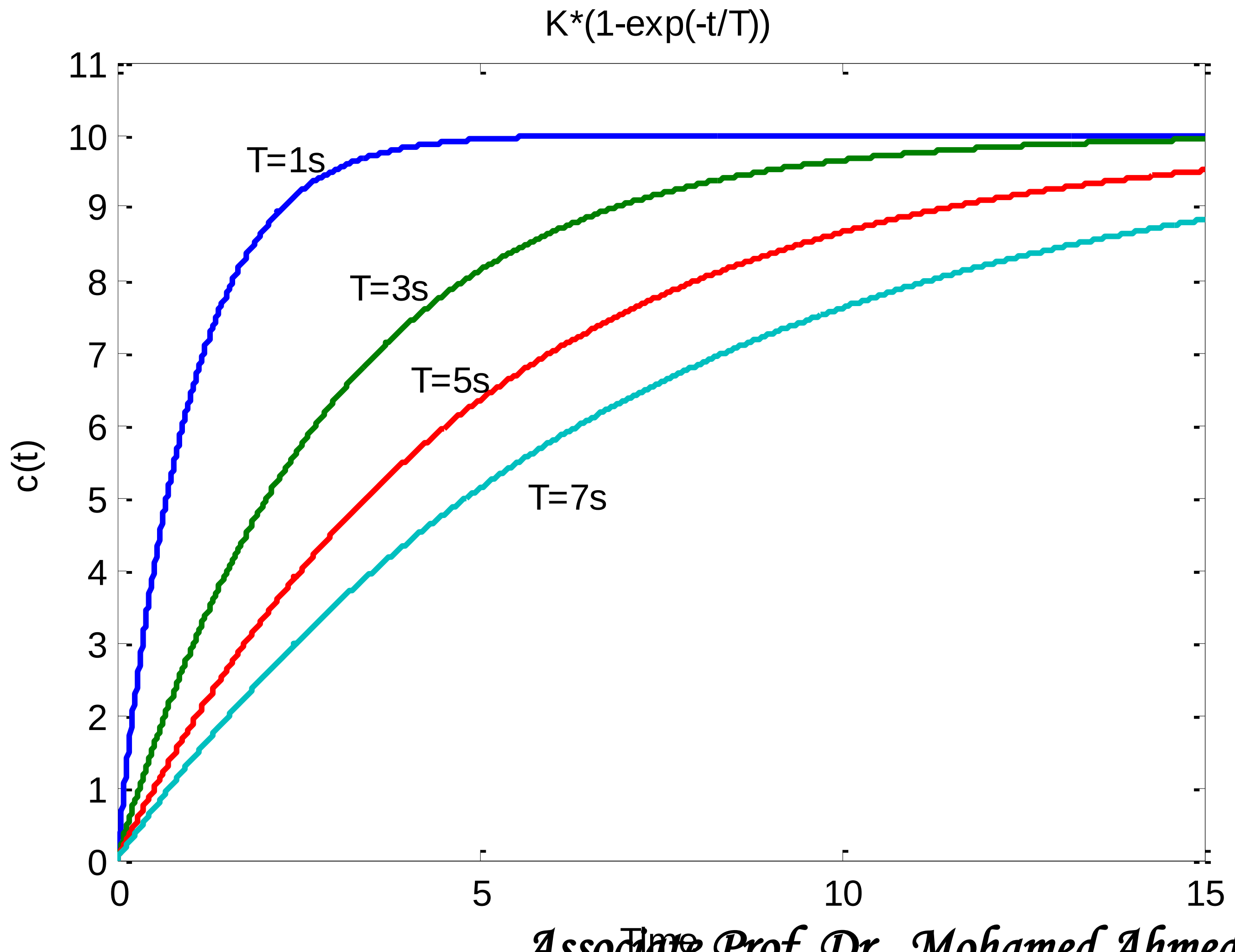
- System takes five time constants to reach its final value.




Ahmed Ebrahim

# Step Response of 1st Order System

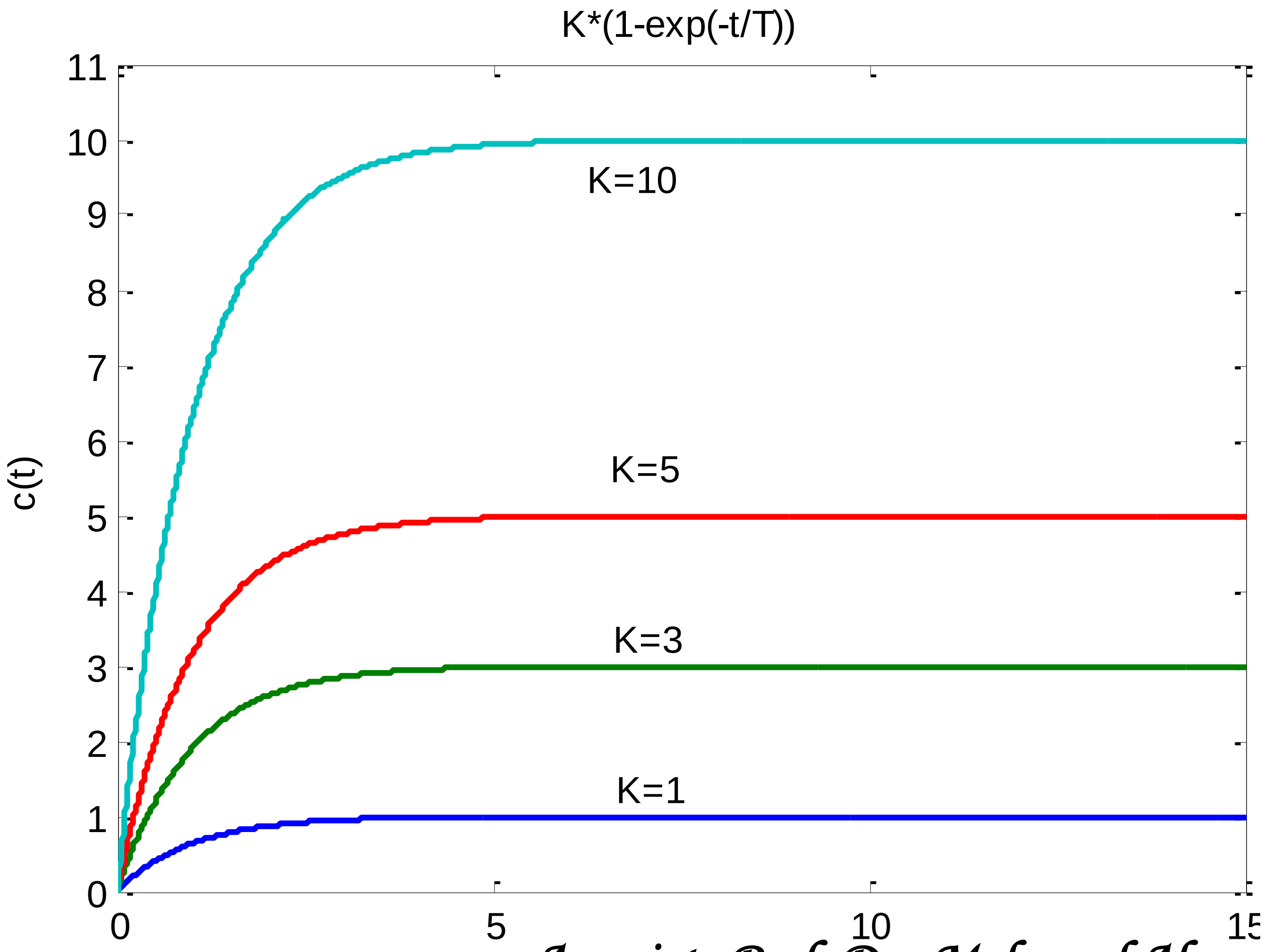
- If K=10 and T=1, 3, 5, 7 $\quad c(t) = K\left(1 - e^{-t/T}\right)$




*Associate Prof. Dr. Mohamed Ahmed Ebrahim*

# Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $$c(t) = K\left(1 - e^{-t/T}\right)$$



# Relation Between Step and impulse response

- The step response of the first order system is

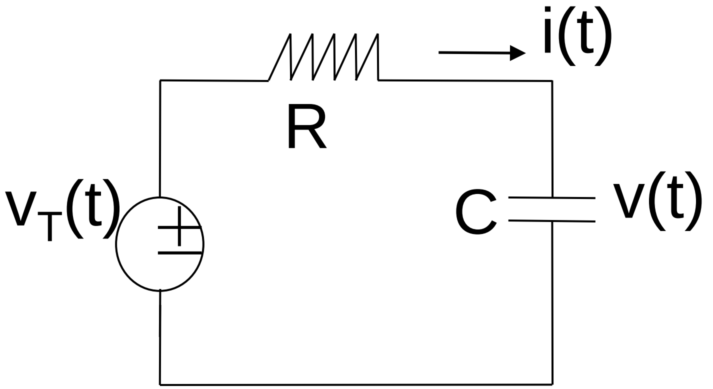
$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

- Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

# Analysis of Simple RC Circuit




$$R \cdot i(t) + v(t) = v_T(t)$$

$$i(t) = \frac{d(Cv(t))}{dt} = C\frac{dv(t)}{dt}$$

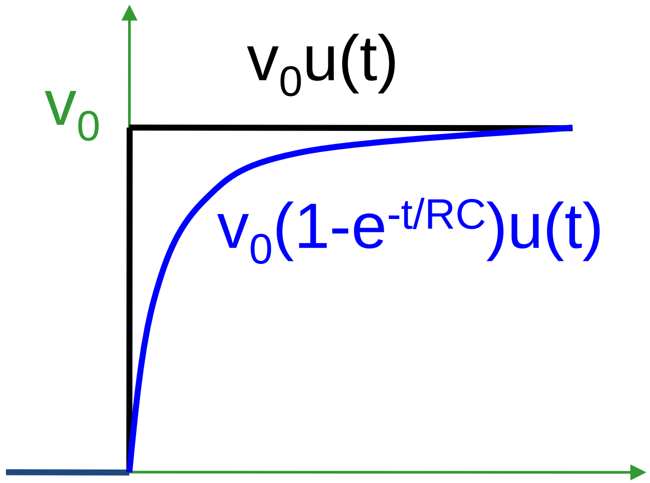
$$\Rightarrow RC\frac{dv(t)}{dt} + v(t) = v_T(t)$$

state variable

Input waveform

# Analysis of Simple RC Circuit

Step-input response:




$$RC\frac{dv(t)}{dt} + v(t) = v_0 u(t)$$

$$v(t) = Ke^{-t/RC} + v_0 u(t)$$

match initial state:

$$v(0) = 0 \quad \Rightarrow K + v_0 u(t) = 0 \Rightarrow K + v_0 = 0$$

output response for step-input:

$$v(t) = v_0(1 - e^{-t/RC})u(t)$$

# Example 1

- Impulse response of a $1^{st}$ order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T
  - D.C Gain K
  - Transfer Function
  - Step Response

# Example 1

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

# Example 1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function $\dfrac{C(s)}{R(s)} = \dfrac{6}{2S+1}$
  - Step Response

# Example 1

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

*Associate Prof. Dr. Mohamed Ahmed Ebrahim*

# Example 1

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

# Ramp Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

- The ramp response is given as

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

# Parabolic Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \quad C(s) = \frac{K}{s^3(Ts+1)}$$

# Practical Determination of Transfer Function of $1^{st}$ Order Systems

- Often it is not possible or practical to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- The system's step response can lead to a representation even though the inner construction is not known.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

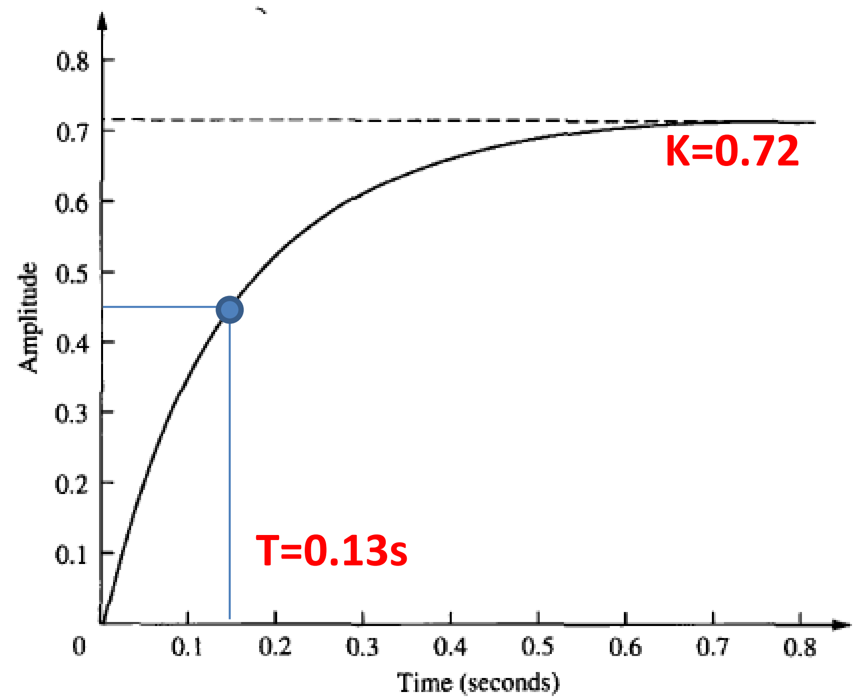
# Practical Determination of Transfer Function of 1st Order Systems

- If we can identify *T* and *K* empirically we can obtain the transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

# Practical Determination of Transfer Function of 1st Order Systems

- For example, assume the unit step response given in figure.
- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.
- K is simply steady state value.




- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

# First Order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

# First Order System With Delays

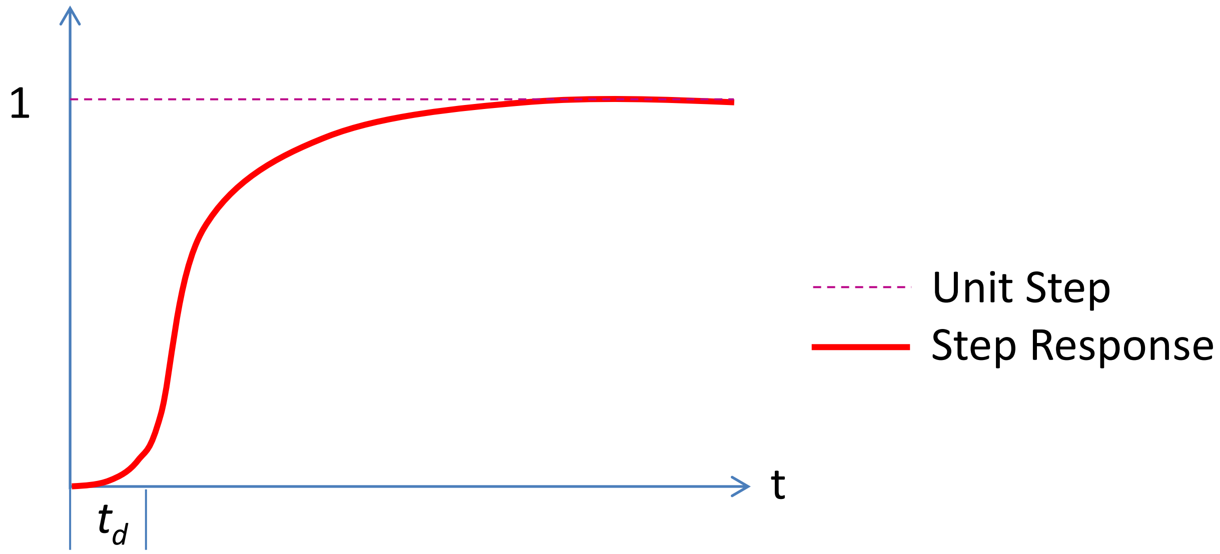
- Following transfer function is the generic representation of 1st order system with time lag.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} e^{-st_d}$$

- Where $t_d$ is the delay time.

# First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} e^{-st_d}$$



# First Order System With Delays

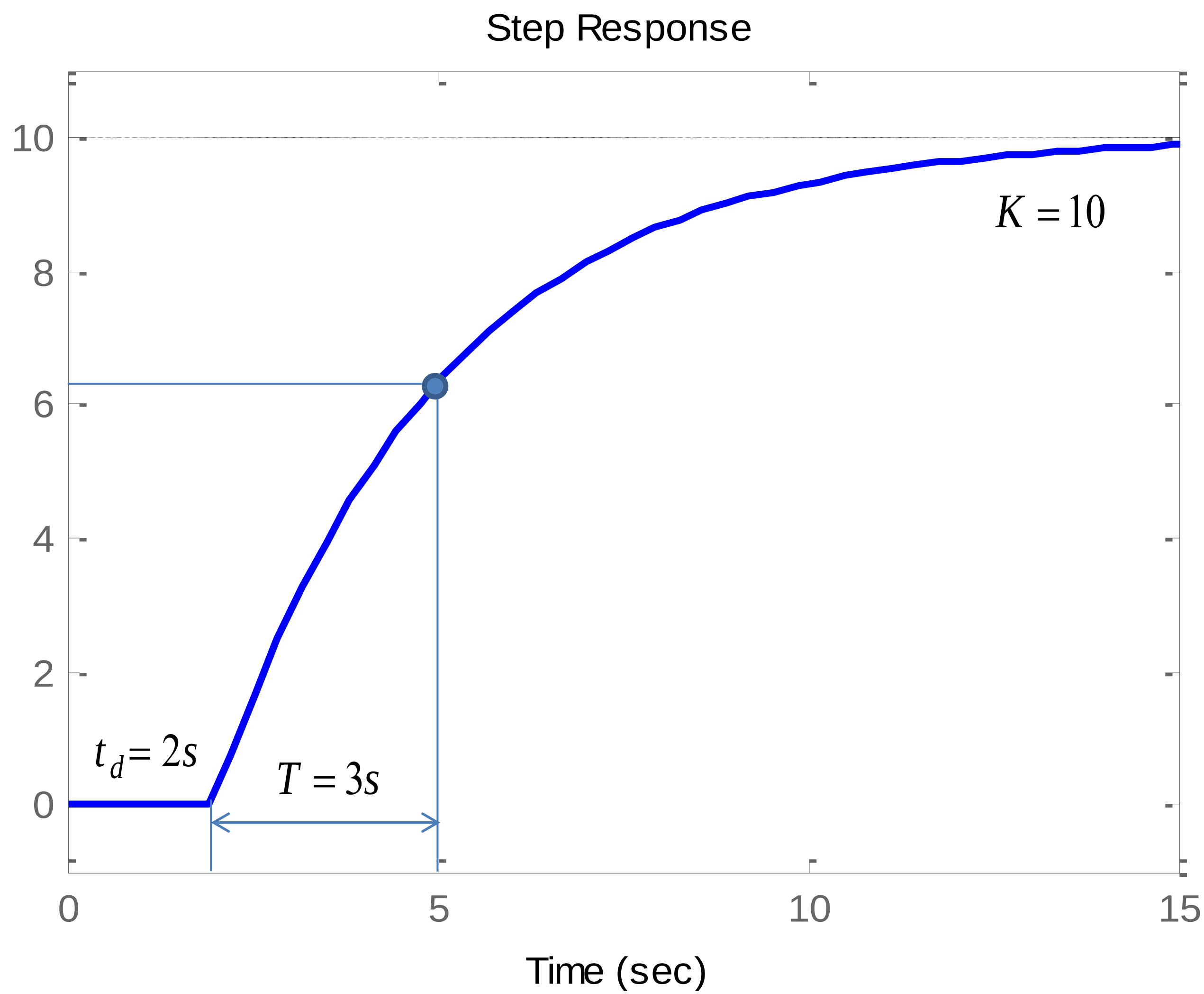
$$\frac{C(s)}{R(s)} = \frac{10}{3s+1}e^{-2s}$$

$$C(s) = \frac{10}{s(3s+1)}e^{-2s}$$

$$L^{-1}[e^{-\partial s}F(s)] = f(t-\partial)u(t-\partial)$$

$$L^{-1}[(\frac{10}{s} + \frac{-10}{s+1/3})e^{-2s}] =$$

$$[10(t-2) - 10e^{-1/3(t-2)}]u(t-2)$$




*Associate Prof. Dr. Mohamed Ahmed Ebrahim*

# Second Order System

- We have already discussed the affect of location of poles and zeros on the transient response of 1$^{st}$ order systems.

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response

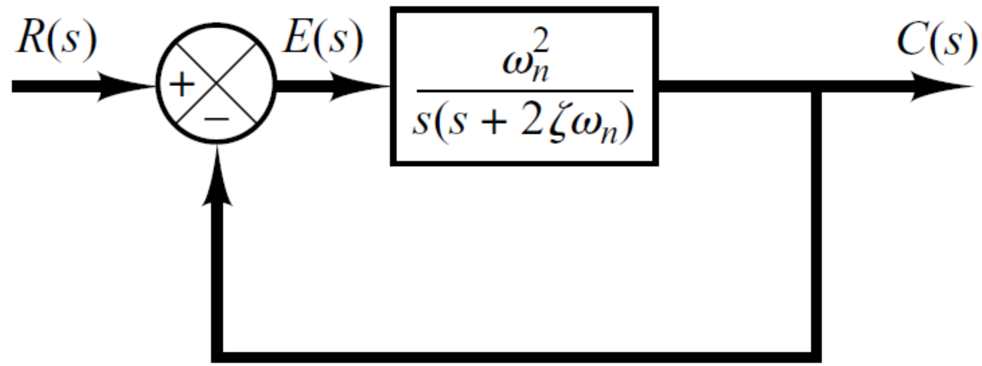
- Whereas changes in the parameters of a second-order system can change the *form of* the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

# Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$




$\omega_n$ $\longrightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta$ $\longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.



*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

# Example 2

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

# Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$


*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

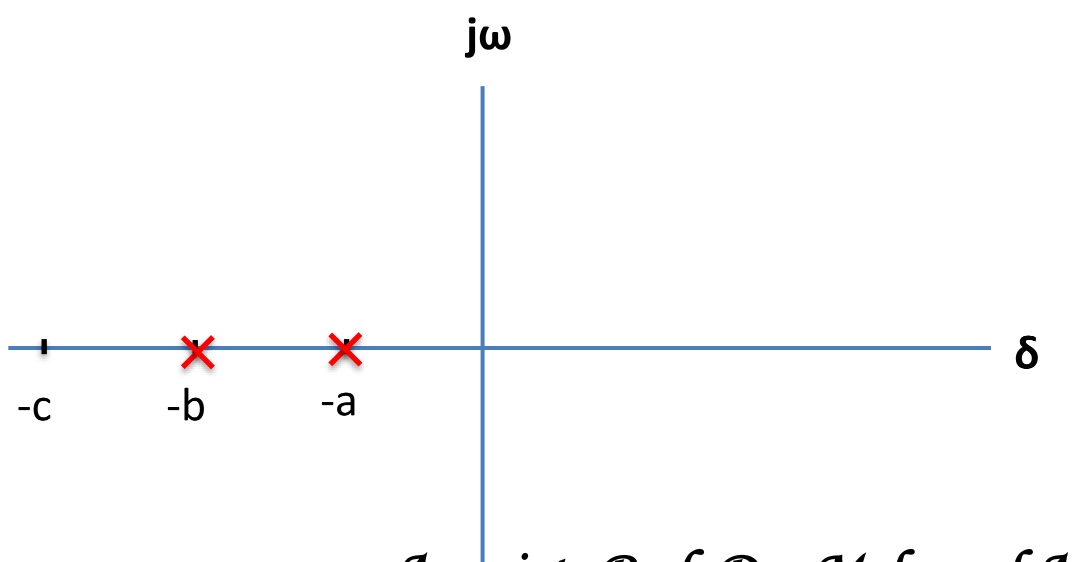
# Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of $\zeta$ , a second-order system can be set into one of the four categories:

  *1. Overdamped* - when the system has two real distinct poles ( $\zeta$ >1).



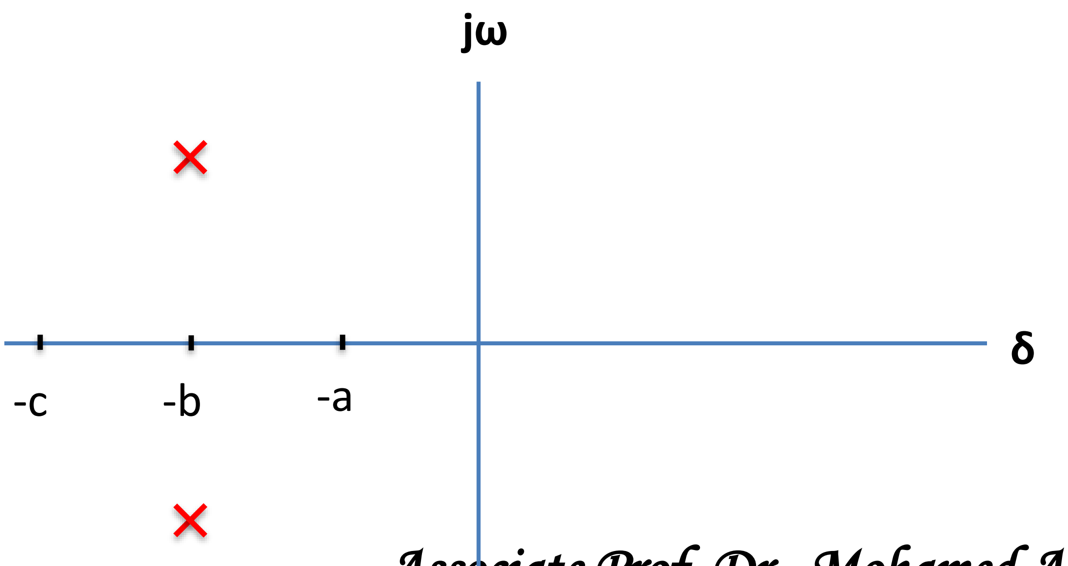
# Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$

$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of $\zeta$, a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



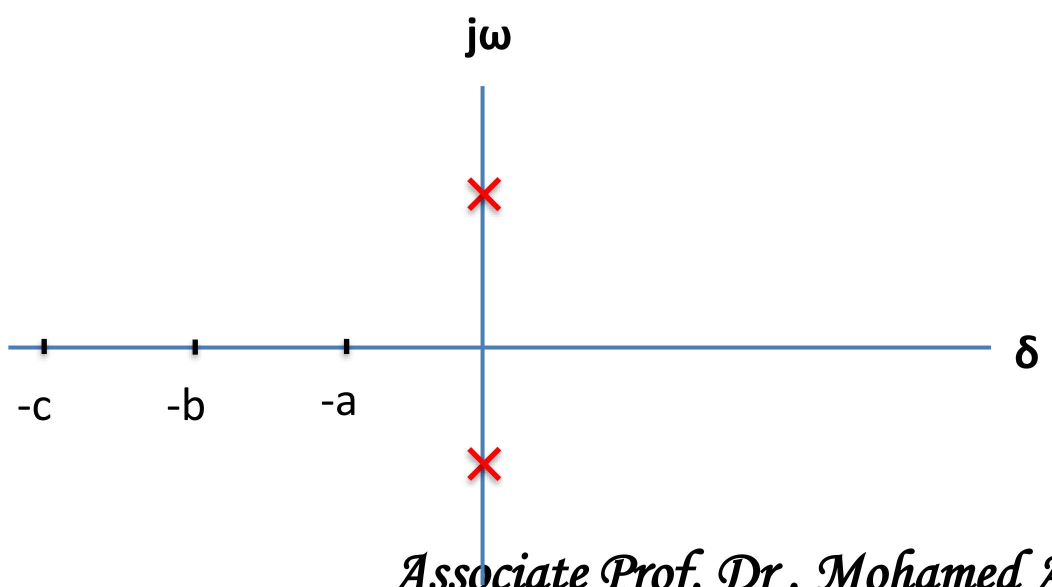
# Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$

$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of $\zeta$, a second-order system can be set into one of the four categories:

3. *Undamped* - when the system has two imaginary poles ( $\zeta$ = 0).

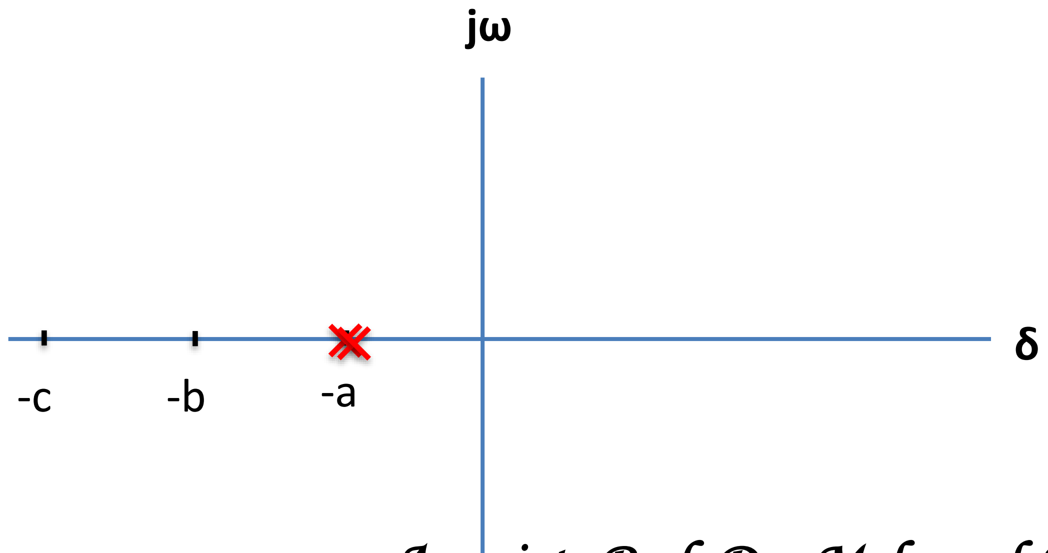


# Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of $\zeta$, a second-order system can be set into one of the four categories:

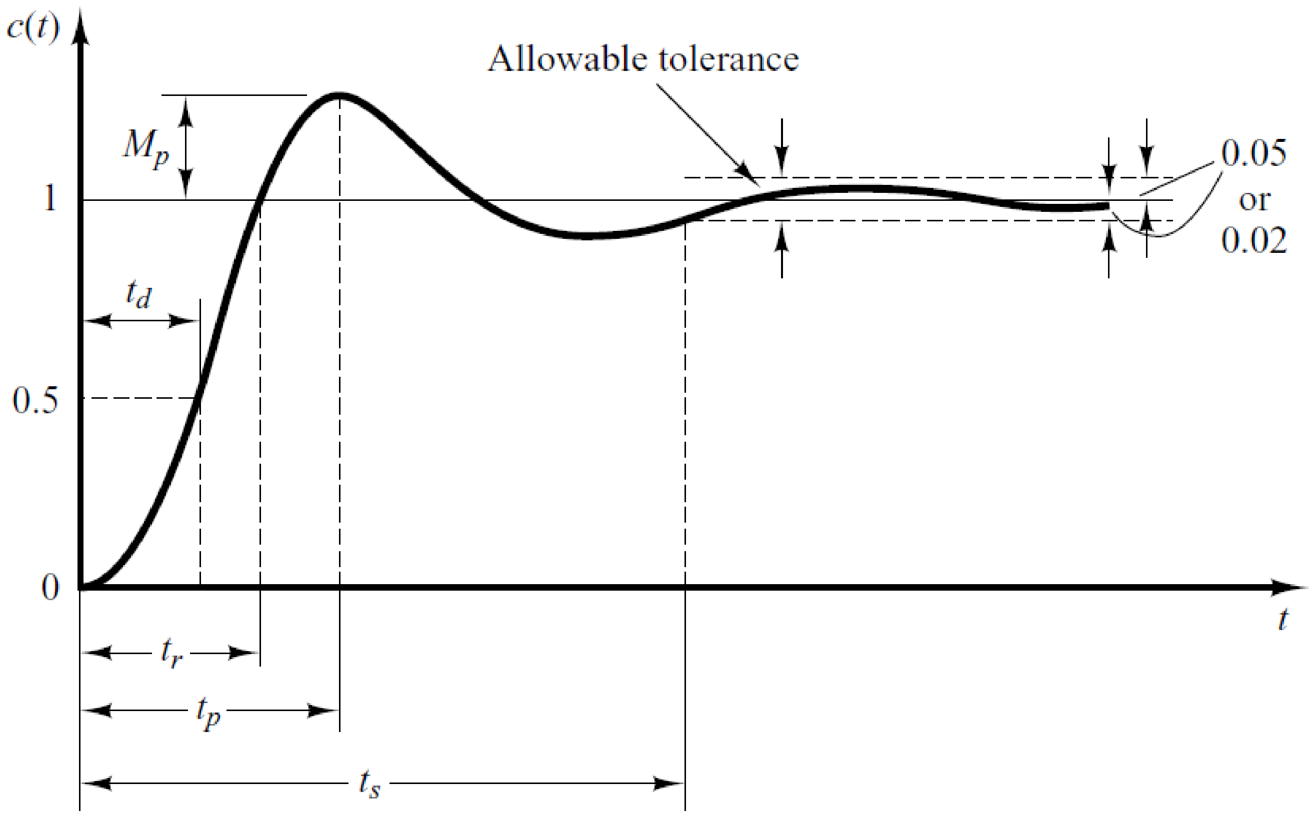
4. *Critically damped* - when the system has two real but equal poles ($\zeta$ = 1).



# Underdamped System

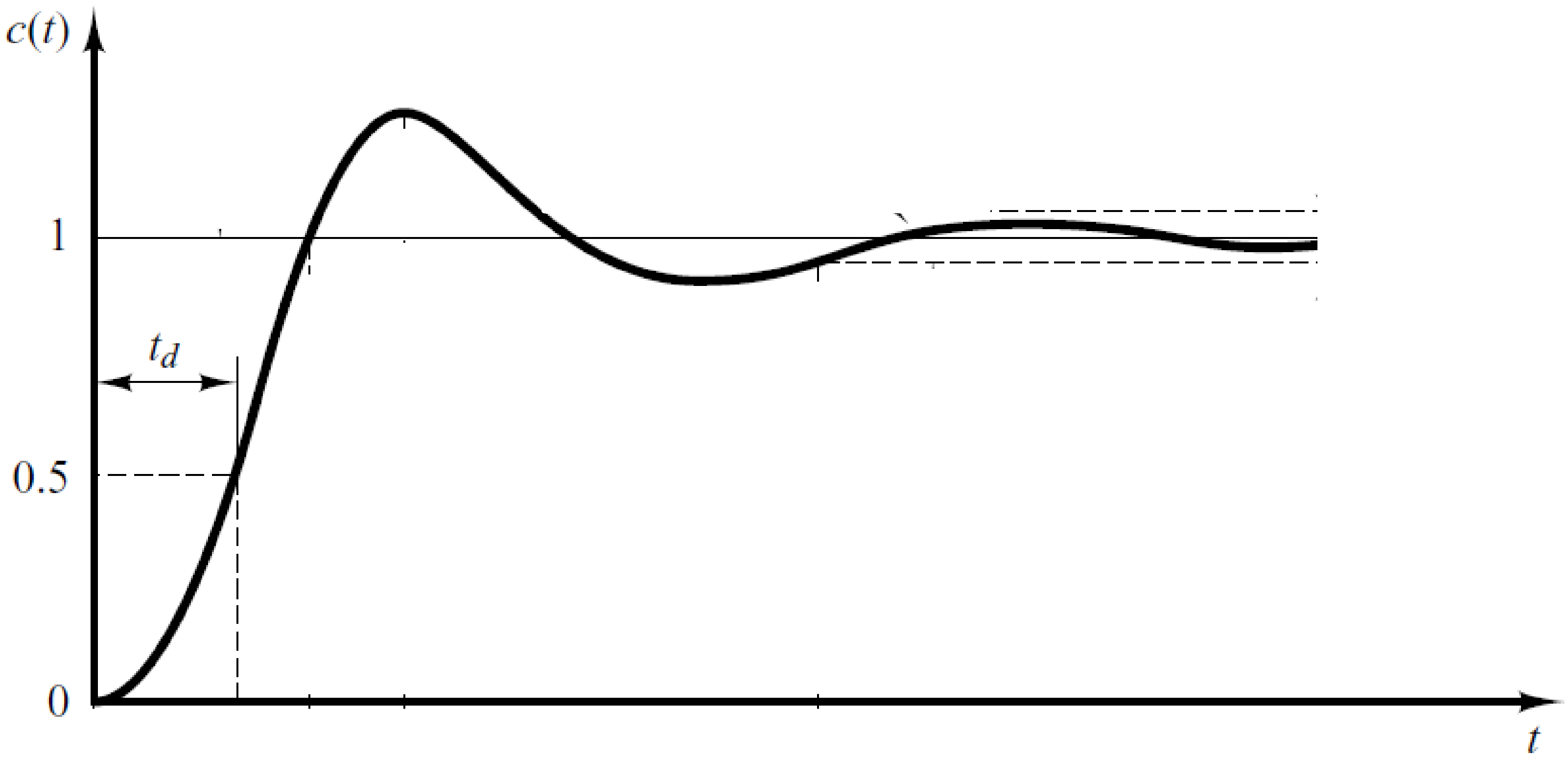
For $0 < \zeta < 1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input is as follows.

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



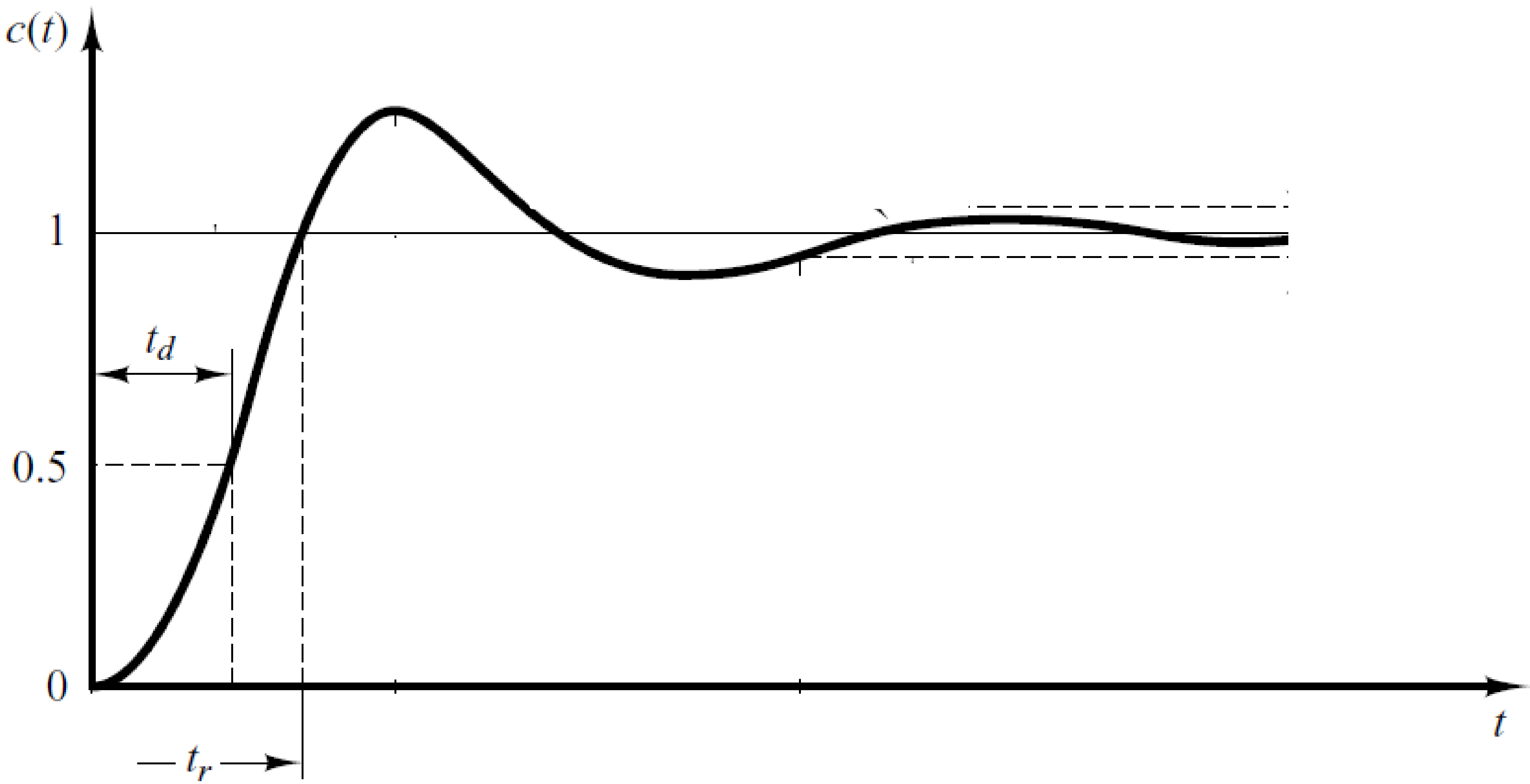
# Delay Time

- The delay ($t_d$) time is the time required for the response to reach half the final value the very first time.



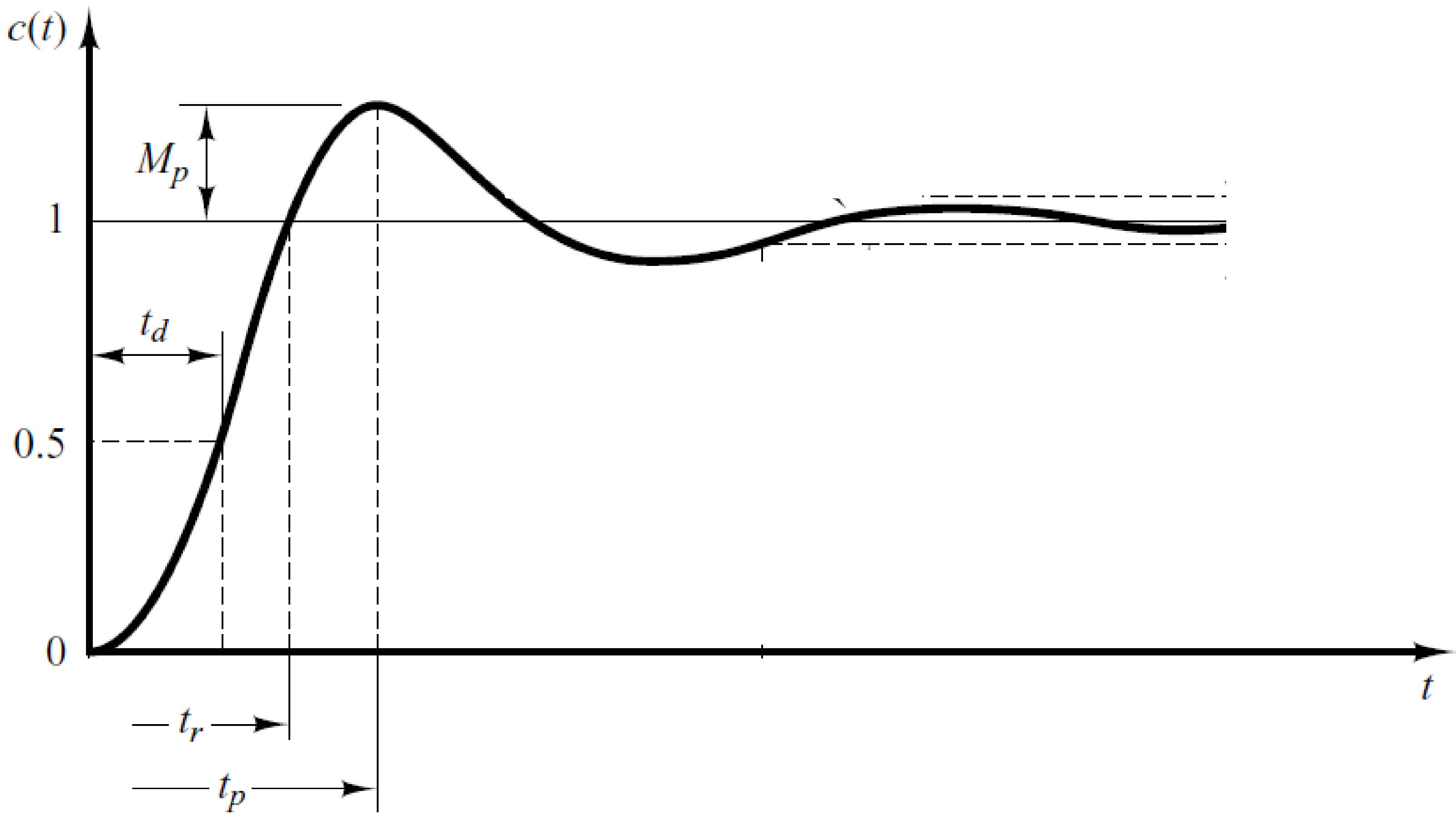
# Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



# Peak Time

- The peak time is the time required for the response to reach the first peak of the overshoot.



# Maximum Overshoot

The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

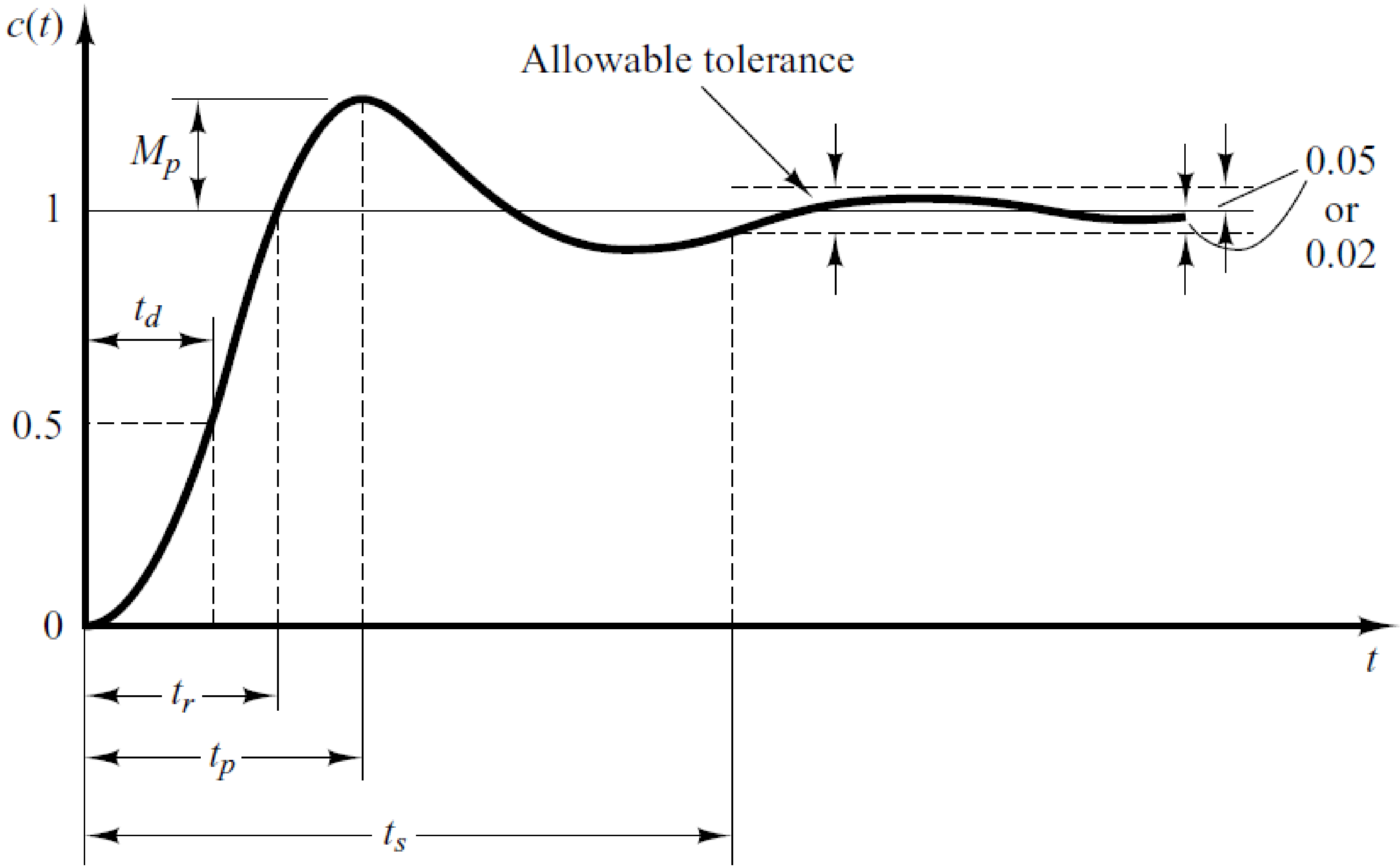
$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.


*Associate Prof. Dr . Mohamed Ahmed Ebrahim*

# Settling Time

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



# Step Response of underdamped System

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\xrightarrow{\text{Step Response}}C(s)=\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n^2\left(1-\zeta^2\right)$$

$$C(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2+\omega_n^2-\zeta^2\omega_n^2}$$

$$(s+2\zeta\omega_n)^2$$

$$C(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2\left(1-\zeta^2\right)}$$

# Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ , is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

# Step Response of underdamped System

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}-\frac{\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}-\frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}$$

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}$$

$$c(t)=1-e^{-\zeta\omega_n t}\cos\omega_d t-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

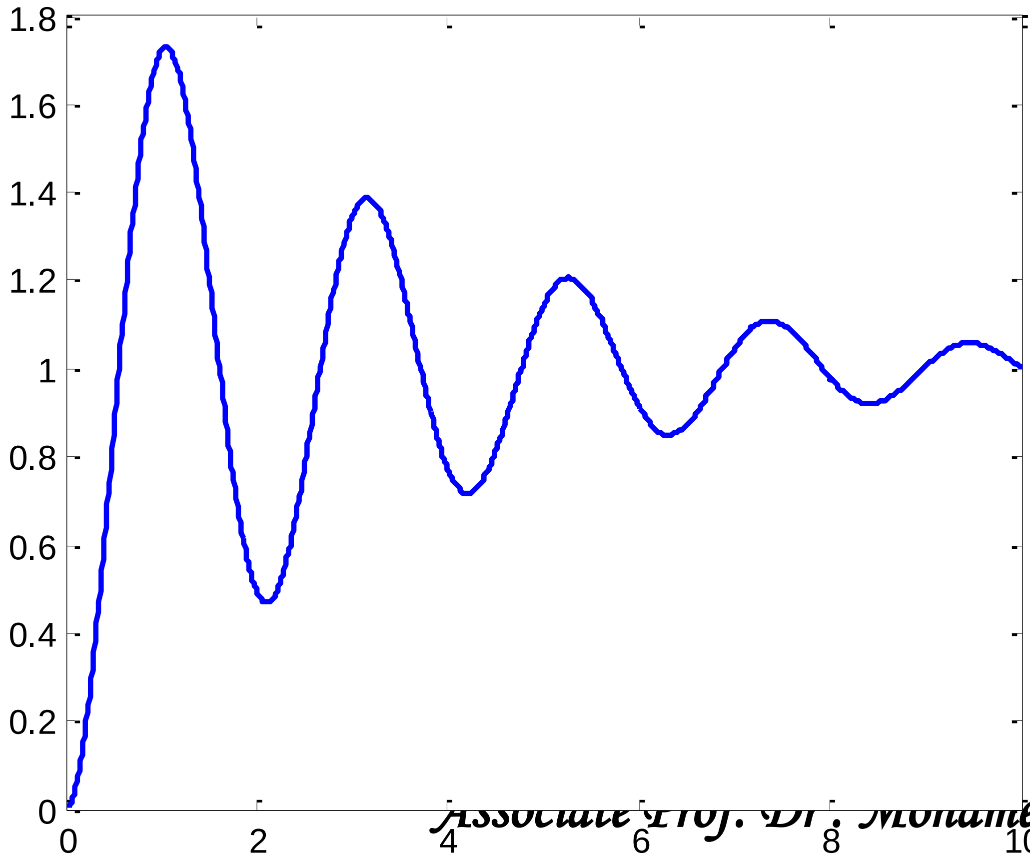
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

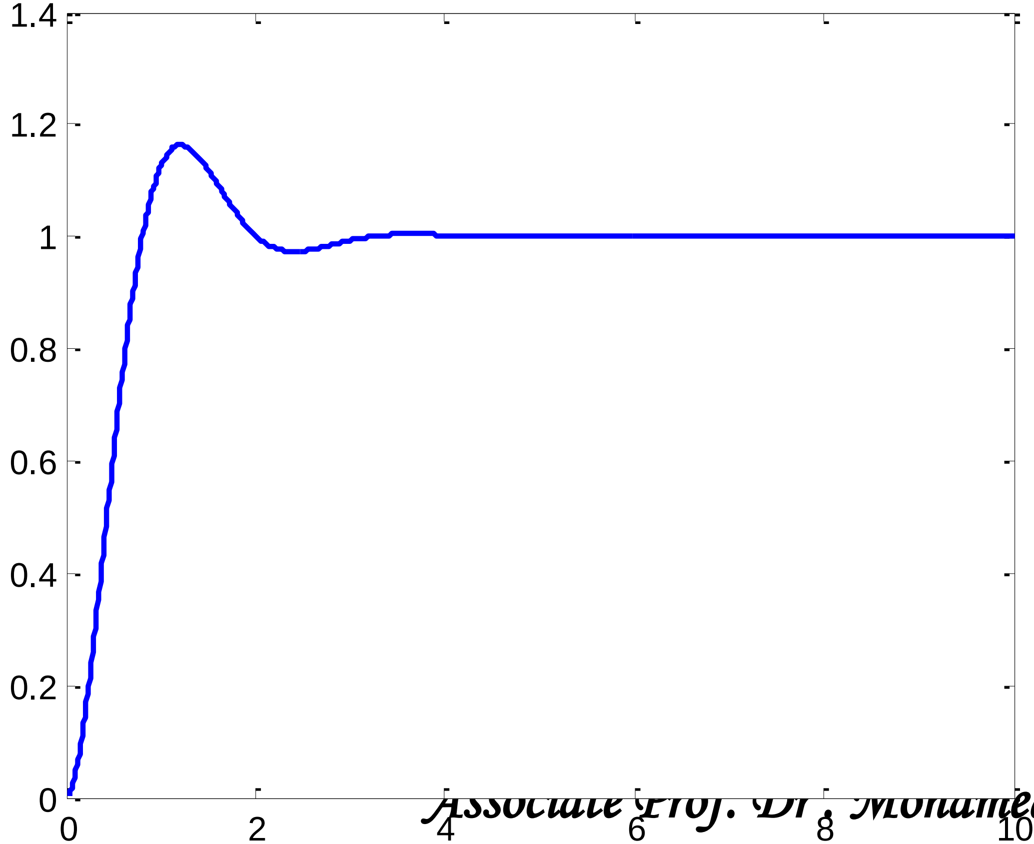
if $\zeta = 0.1$ and $\omega_n = 3$



# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

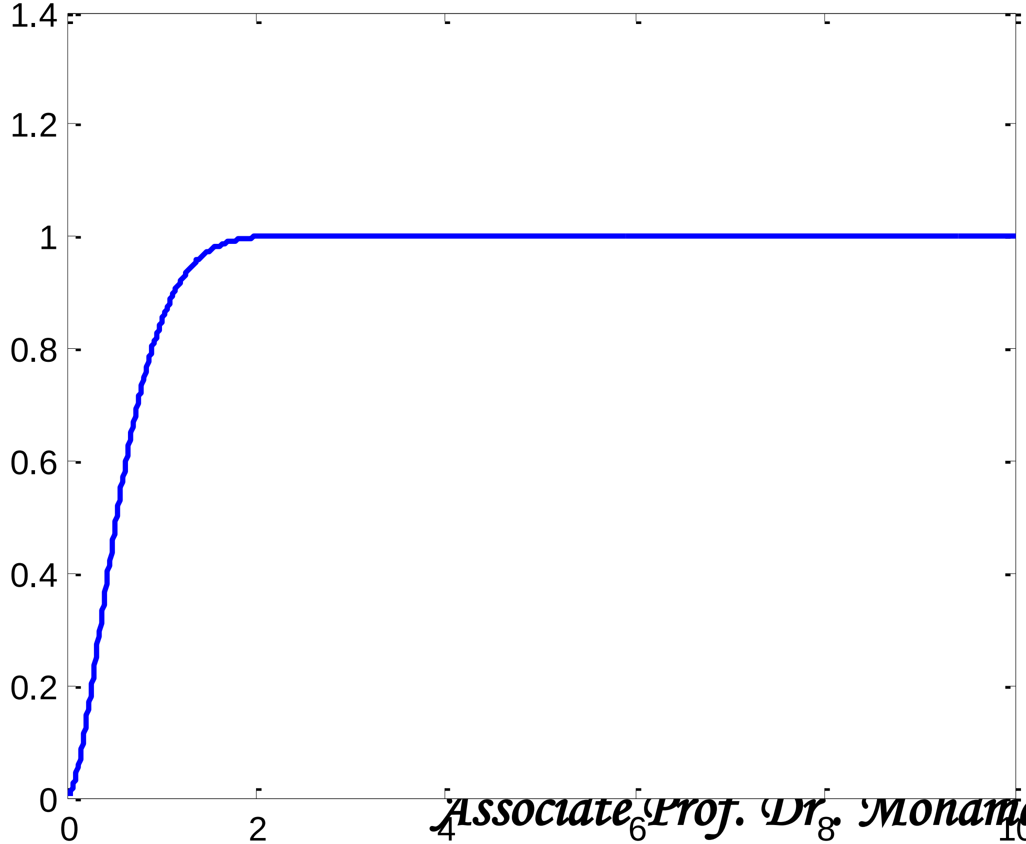
if $\zeta = 0.5$ and $\omega_n = 3$



# Step Response of underdamped System

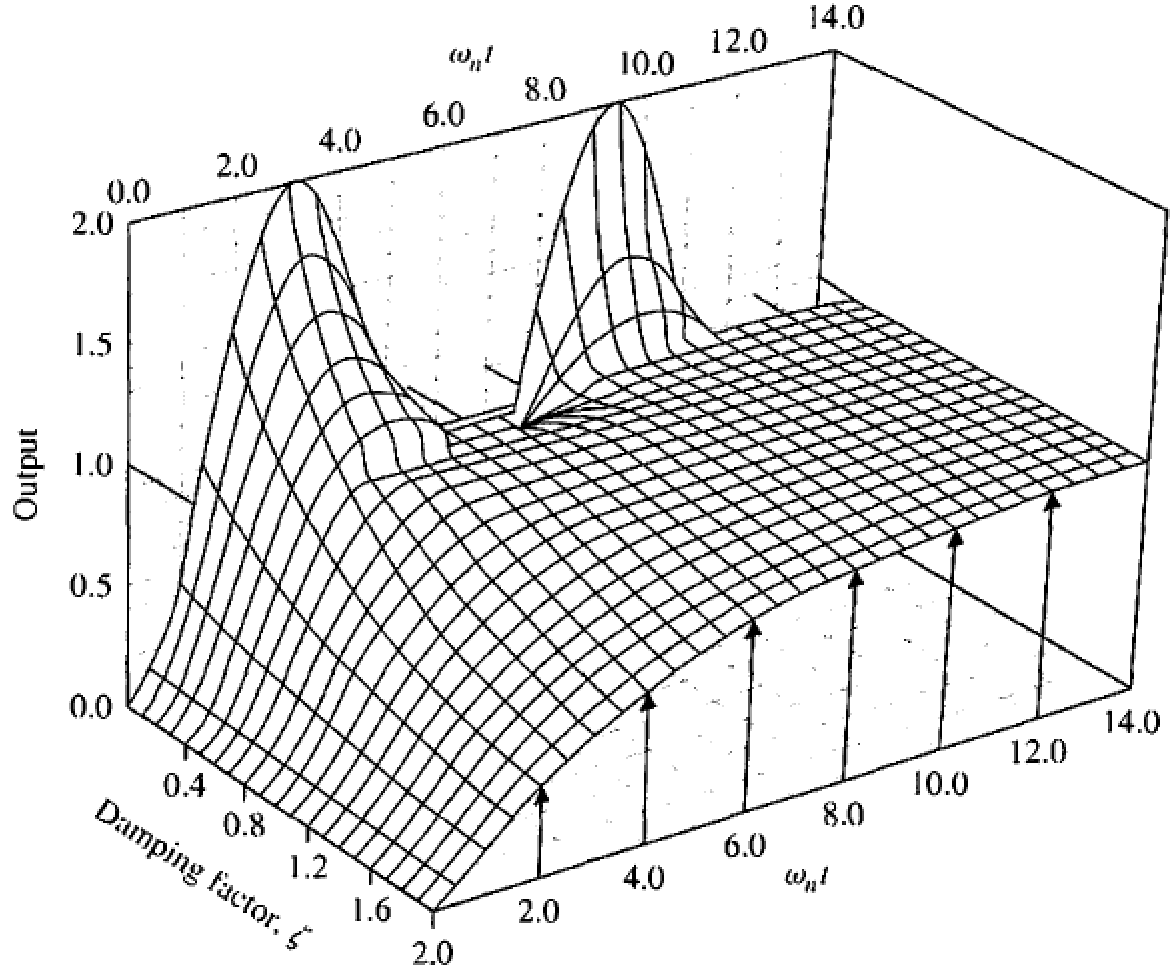
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3$



# Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

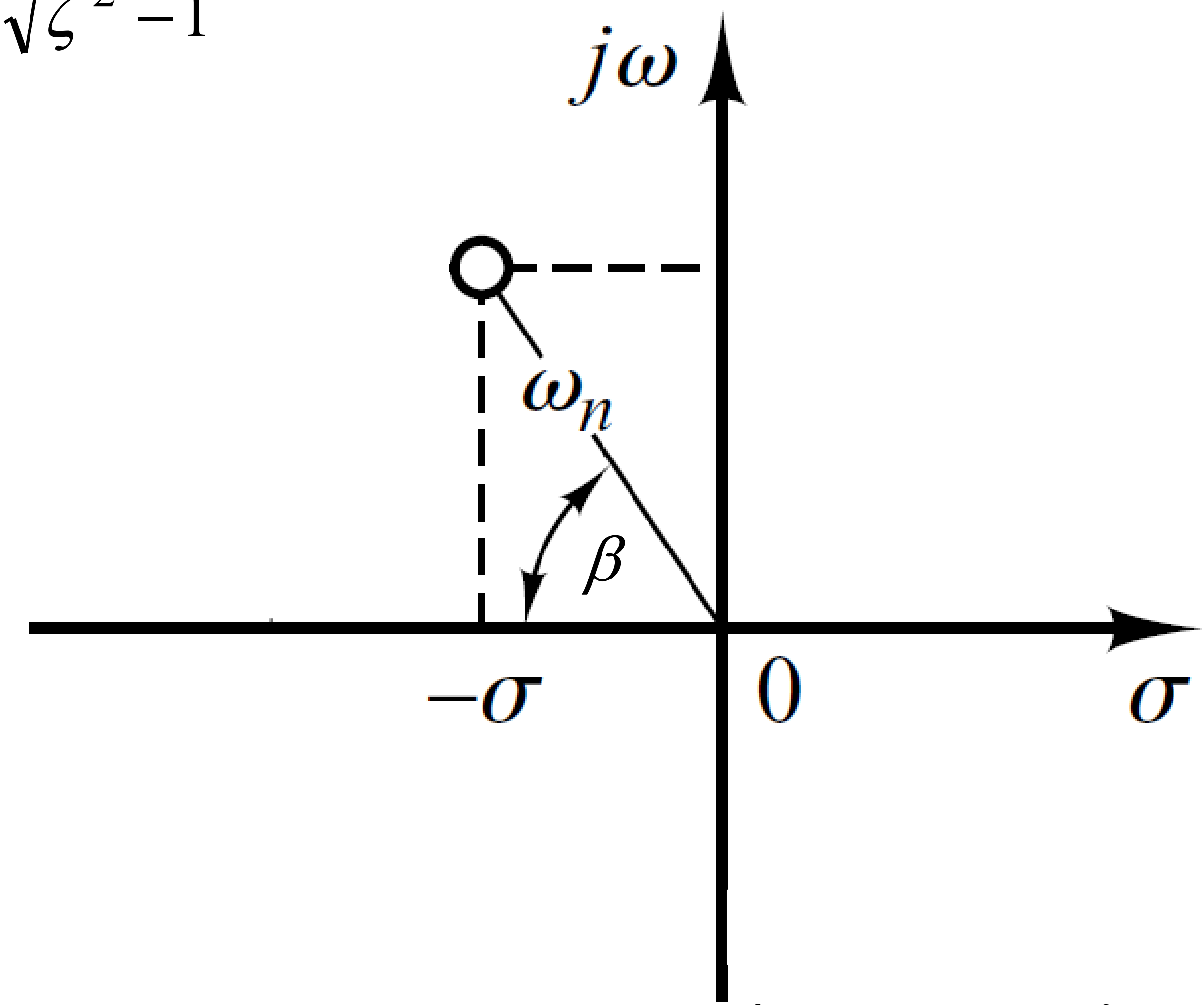



med Ebrahim

# S-Plane (Underdamped System)

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Since $\omega^2\zeta^2 - \omega^2(\zeta^2 - 1) = \omega^2$, the distance from the pole to the origin is $\omega$ and $\zeta = cos\beta$





ed Ebrahim

# Analytical Solution

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- Rise time: set c(t)=1, we have $t_r = \frac{\pi-\beta}{\omega_d}$ $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$
- Peak time: set $\frac{dc(t)}{dt} = 0$, we have $t_p = \frac{\pi}{\omega_d}$
- Maximum overshoot: $\mathrm{M}_p = c\left(t_p\right) - 1$ $= e^{-(\zeta\omega/\omega_d)\pi}$ (for unity output)
- Settling time: the time for the outputs always within 2% of the final value is approximately $\frac{4}{\zeta\omega}$

# *With Our Best Wishes*
# *Automatic Control (1)*
# *Course Staff*

# Thank You
# For Your Attention

**Mohamed Ahmed Ebrahim**

*Associate Prof. Dr . Mohamed Ahmed Ebrahim*